Stability of postulated, self-similar, hydrodynamic blowup solutions

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(Received 8 June 2000)

A solution with real time singularity is assumed to exist that is steady under a Leray-type normalization. This solution is further assumed to be reached asymptotically as $t \rightarrow t_0$ in the renormalized plane, and thus can be thought of as the leading behavior of an inner solution. Constraints due to conserved quantities like energy are shown to be weakened in this scenario. In the wake region that trails the collapsing structure, it is shown that eigenfunctions associated with initial conditions are stable and decay, allowing the attracting singular solution to be shielded from details of the initial conditions. The parameters of the normalization are t_0, r_0, v_0, λ, and α, which are the critical time, the location of the singularity, the velocity of the singular point, a scaling factor, and the scaling exponent of the velocity $(t_0 - t)^{\alpha}$. The stability of the eigenfunctions of this solution obtained from the perturbation of these parameters is also examined in this work. Perturbations in the critical time and location are shown to be unstable whereas perturbations in velocity and scaling are not. The condition that the amplitude of the unstable eigenfunctions vanishes determines the time and location of the singularity.

PACS number(s): 47.20.−k, 47.10.+g, 47.15.Ki

I. INTRODUCTION

Numerical evidence from Grauer, Marliani, and Germaschewski [1], Pelz and Gulak [2], Pelz [3], Boratav and Pelz [4], and Kerr [5] suggests that vortex collapse solutions of the equations of incompressible inviscid flow exhibit a singularity in real time. Whether a smooth initial flow develops a singularity spontaneously in a finite time is a fundamental question, since it signals a breakdown of the equations of motion.

As was suggested by Leray [6], one possible route to blowup that is amenable to analysis and computation is self-similar collapse. The description of a spherically symmetric collapse should identify a combination of radial and time variables that is fixed in a collapsing wave structure that reaches the origin in finite time. If the evolution of the collapse has self-similarity, the solution will appear steady under a suitable change of variables. Greene and Boratav [7] proposed such a renormalization based on the inverse time, pointwise blowup of vorticity result of Beale, Kato, and Majda [8]. While the blowup solutions in general, and self-similar ones in particular, have not been proved to exist, we shall assume optimistically that they do in this paper, and proceed to study the consequences of this assumption.

The structure of this kind of self-similar solution is limited, however. Nečas [9] showed for the Navier-Stokes equations that there are no global Leray-type self-similarity solutions $v(x,t) = \lambda V[x/(\lambda\sqrt{t_0 - t})]/\sqrt{t_0 - t}$, $V \epsilon L^3(\mathbb{R}^3)$, where t_0 is the critical time and λ is a scaling constant. Tsai [10] extended this work for certain types of local self-similarity solution. Constantin [11] commented that inviscid flows have very restricted scaling for global self-similarity.

Despite these restrictions, *local* self-similarity may exist as the leading behavior of an inner solution to the Euler equations. This solution would be approached asymptotically as a critical time is approached and in the inner (renormalized) variables. Nonzero outer boundary conditions, which in the global case lead to unbounded energies, for the inner problem can yield finite energy.

Under such conditions, the structure of the solution can be analyzed. In this paper, we assume that a local self-similar solution exists and that under the associated renormalization a steady solution exists. We analyze how conservation laws affect such solutions. We show that the collapse solution can be attracting by showing that a large class of eigenfunctions associated with initial conditions are stable and decay. We examine the linear stability of the steady renormalized flow in terms of eigenfunctions associated with the parameters of the normalization. That the collapse solution be stable and attracting is important for realizability.

In particular, in Sec. II we review the renormalization and introduce the linearized perturbation problem. In Sec. III we apply known conservation laws to the solution in the transformed plane to derive constraints. In Sec. IV we address the issue of an attracting solution by showing that modes from the initial conditions decay. In Sec. V we examine those eigenfunctions stemming from perturbations in the parameters of the scaling transformation.

II. THE SCALING TRANSFORMATION

For completeness, we repeat some of the presentation of Greene and Boratav [7]. We begin with the Euler equations for incompressible flow in rotation form,

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$$\frac{\partial v}{\partial t}=v\times\omega-\nabla\Pi, \tag{2.1}$$

where v is the Eulerian representation of the fluid velocity (which is solenoidal, $\nabla\cdot v=0$), $\omega=\nabla\times v$ is the vorticity, and $\Pi=p/\rho+v^2/2$ is the stagnation pressure with the density $\rho=1$. The stagnation pressure is determined through the condition that incompressibility is preserved by the flow,

$$\Delta\Pi=\nabla\cdot(v\times\omega), \tag{2.2}$$

together with boundary conditions. Here they are chosen to be consistent with a configuration in which the magnitudes and directions of the coarse-grained averages of the velocity and vorticity fields fall off rapidly toward infinity.

We now assume that a singular collapse occurs at some time t_0, location in the interior of the flow r_0, and velocity v_0. In order to follow the development of this collapse, we introduce the scaling transformation $(r,t)\rightarrow(\xi,\tau)$, appropriate for an amplifying, spherically symmetrical collapsing wave,

$$\xi=\frac{r-r_0+v_0(t_0-t)}{\lambda_0(t_0-t)^{1-\alpha}},$$

$$\tau=t-t_0,$$

$$v=\frac{\lambda_0}{(t_0-t)^{\alpha}}V(\xi,\tau)+v_0,$$

$$\omega=\frac{1}{t_0-t}\nabla_\xi\times V(\xi,\tau),$$

$$\Pi=\frac{\lambda_0^2}{(t_0-t)^{2\alpha}}P(\xi,\tau)+\frac{\lambda_0}{(t_0-t)^{\alpha}}v_0\cdot V(\xi,\tau). \tag{2.3}$$

The quantity α is a scaling parameter that specifies the relative degree to which the wave is amplified as it is compressed. This transformation is similar to that proposed by Leray [6] for the Navier-Stokes equations; however, he found dissipation constrains α to be one-half. The parameter λ_0 represents invariance under choice of length scale; note that λ_0 and α always occur together in the combination $\lambda_0(t_0-t)^{-\alpha}$. Here ξ, V, and P are nondimensional when $\lambda_0(t_0-t)^{-\alpha}$ is assigned the dimension of a velocity. The set of equations (2.3) is a simple transformation of variables from (r,t,v,Π) to (ξ,τ,V,P) that contains no information about the evolution of Eqs. (2.1) and (2.2). While the solutions are not affected by this transformation of variables, the volume in phase space occupied by a set of solutions need not be invariant. Thus the flow in function space of Eq. (2.1) can be Hamiltonian while the flow of Eq. (2.4) is attracting.

Applying the renormalization Eq. (2.3) to Eq. (2.1) we find that the evolution in the transformed variables is given by

$$\tau\frac{\partial}{\partial\tau}V=\alpha V+(1-\alpha)(\xi\cdot\nabla_\xi)V-V\times(\nabla_\xi\times V)+\nabla_\xi P. \tag{2.4}$$

This result is independent of (r_0,t_0,v_0,λ_0) since the parameters in this transformation reflect invariance properties of Eqs. (2.1) and (2.2).

Rearranging, we arrive at the form that we shall use,

$$\tau\frac{\partial}{\partial\tau}V=(2\alpha-1)V-[V+(1-\alpha)\xi]\times(\nabla_\xi\times V)+\nabla_\xi[P+(1-\alpha)\xi\cdot V], \tag{2.5}$$

with Eq. (2.2) becoming

$$\Delta_\xi P=\nabla_\xi\cdot[V\times(\nabla_\xi\times V)]. \tag{2.6}$$

Note that the right side of Eq. (2.5) has no explicit dependence on τ. Thus, if the right side vanishes, it vanishes for all τ, and, from Eq. (2.3), the evolution of Eqs. (2.1) and (2.2) is singular for positive α. Steady solutions for which the right side of Eq. (2.5) vanishes will be denoted $V_0(\xi),P_0(\xi)$. Singularity of the Euler flow follows from the existence of steady solutions and does not depend on the algorithm that yielded this state nor on this steady state being an attractor of the flow of (2.5). As discussed in the Introduction, Eq. (2.5) is valid in the inner region, and outer boundary conditions $(\xi\rightarrow\infty)$, which could be on a spherically collapsing ball, should match the inner conditions of an outer solution.

The status of this flow as an attractor under the dynamics of Eq. (2.1) can be assessed by evaluating eigenfunctions and eigenvalues obtained from linearization of the equations with respect to small perturbations off the base solution (V_0,P_0). Setting

$$V=V_0+\epsilon V', \quad P=P_0+\epsilon P', \tag{2.7}$$

substituting Eq. (2.7) into Eqs. (2.5) and (2.6), and linearizing, we find that V' and P' satisfy

$$\tau\frac{\partial}{\partial\tau}V'=(2\alpha-1)V'-[V_0+(1-\alpha)\xi]\times(\nabla_\xi\times V')-V'\times(\nabla_\xi\times V_0)+\nabla_\xi[P'+(1-\alpha)\xi\cdot V'] \tag{2.8}$$

and

$$\Delta_\xi P'=\nabla_\xi\cdot[V'\times(\nabla_\xi\times V_0)]+\nabla_\xi\cdot[V_0\times(\nabla_\xi\times V')]. \tag{2.9}$$

Since the right side of these equations is not explicitly τ dependent, the time dependence can be treated by separation of variables. Note that separation of variables can be employed on such linear, homogeneous equations with homogeneous boundary conditions. Denoting the separation constant by q and using an obvious notation to distinguish the time-dependent portion of the eigenfunction, we are led to

$$V'(\xi,\tau)=(-\tau)^{-q}\delta V(\xi) \tag{2.10}$$

and

$$0=(q+2\alpha-1)\delta V-[V_0+(1-\alpha)\xi]\times(\nabla_\xi\times\delta V)-\delta V\times(\nabla_\xi\times V_0)+\nabla_\xi[\delta P+(1-\alpha)\xi\cdot\delta V], \tag{2.11}$$

so that q is an eigenvalue. Perturbations grow if the real part of q is positive for $\tau\rightarrow 0$. The behavior of the full basis of eigenfunctions is necessary to assess the full stability of the steady solution (V_0, P_0).

Since V_0 and P_0 are not known specifically, it is impossible to perform a complete stability analysis. We can, however, examine the growth of a certain subset of modes associated with the parameters of the scaling transformation (2.3). This analysis is presented in Sec. V.

III. CONSTRAINTS FROM CONSERVATION LAWS

At this stage in the development we have two coordinate systems for treating the evolution, one fixed and the other contracting, together with differential equations to be satisfied. We can expect that certain further constraints in the form of initial and boundary conditions will be required. In this section the role played by conservation laws in supplying acceptability criteria is considered. In particular, this includes criteria for the choice of α in Eq. (2.3).

We assume that there are nonsingular initial conditions for Eqs. (2.1) and (2.2) that asymptotically show self-similar collapse. By this we mean that, in the limit as τ approaches zero, V and P evolve to a steady solution to Eqs. (2.5) and (2.6). Thus V_0 is defined as a limit, and the vanishing of the right side of Eq. (2.5) is asymptotic in the limit of vanishing τ. That is,

$$V_0(\xi)=\lim_{\tau\rightarrow 0} V(\xi,\tau). \tag{3.1}$$

Note that the freedom of $V(\xi,\tau)$ associated with the choice of initial conditions is lost for V_0 since the latter must satisfy the steady version of Eq. (2.5). Thus information on initial conditions is completely lost in finite time in taking the limit of Eq. (3.1). It follows that extreme care is required in extracting information from conservation laws if corruption of this information is to be avoided.

In a physical problem with conserved energy, such as this, the form of the conserved energy generally plays a fundamental role. From Eq. (2.1) we see that the conserved energy is the volume integral of the square of the velocity in the interior of a sphere, in the limit that the sphere is infinite. Transforming this to the scaled variables of Eq. (2.3), we find for the kinetic energy E

$$E=\lim_{r_b\rightarrow\infty}\int_{|r|<r_b}\tfrac{1}{2}v^2 dr=\lambda_0^5(t_0-t)^{3-5\alpha}\lim_{\xi_b\rightarrow\infty}\int_{|\xi|<\xi_b}\tfrac{1}{2}V^2 d\xi, \tag{3.2}$$

where

$$\xi_b=\frac{r_b}{\lambda_0(t_0-t)^{1-\alpha}}. \tag{3.3}$$

The time-dependent factor on the right is a measure of the energy that is left behind in the wake of a collapsing wave. Since V_0 is a quantity of interest, it would be useful to replace the scaled velocity V with V_0 defined in Eq. (3.1) as a limit $t\rightarrow t_0$. Since energy is conserved, the last expression in Eq. (3.2) is independent of time and thus appears to be amenable to the introduction of the limit of Eq. (3.1). However, there are subtleties in taking the limit of (t_0-t) vanishing while ξ_b or r_b becomes infinite. In particular, note that ξ_b can approach infinity while r_b remains small. Thus the limiting volume can be quite different depending on the order of the limits of large r_b and small t_0-t. If the solution is self-similar in all of R^3, then energy conservation requires $\alpha=3/5$, a result suggested by Constantin [11] (the length scaling is $1-\alpha$ or 2/5) and by Pomeau [12]. However, we can define V_0 as a limit holding ξ fixed, and thus it is defined only in a region of r that has asymptotically vanishing volume in the limit that $t-t_0$ vanishes. Information lying in the region of finite r, where the energy resides, is lost in the limit of vanishing τ. Therefore, in the case of asymptotic self-similarity nothing can be learned about V_0 from the conservation of energy.

There are other conservation laws leading to different constraints on the solutions and thus to different plausible choices of the value of α. From conservation of circulation, we arrive at a scaling of $\alpha=1/2$ without limiting procedures. Indeed,

$$\Gamma=\oint v\cdot dr=\lambda_0^2(t_0-t)^{1-2\alpha}\oint V\cdot d\xi. \tag{3.4}$$

The invariance of this quantity is related to the Helmholtz conservation laws.

Closely related to the circulation is the helicity. Thus if we define the conserved quantity

$$H=\lim_{r_b\rightarrow\infty}\int_{|r|<r_b} v\cdot\omega\, dr$$
$$=\lambda_0^4(t_0-t)^{2-4\alpha}\lim_{\xi_b\rightarrow\infty}\int_{|\xi|<\xi_b} V\cdot\nabla_\xi\times V\, d\xi \tag{3.5}$$

in certain regimes, e.g., compact vorticity, this leads to $\alpha=1/2$ as a preferred value. Greene and Boratav [7] showed that the numerical results of [4] strongly favored $\alpha=1/2$.

Finally, consider the conservation of mass. As with the energy, Eq. (3.2), mass must be left behind in the wake of a collapsing wave. Equation (2.5) contains terms that are driven by the necessity of satisfying this condition quantitatively.

IV. INDEPENDENCE FROM INITIAL CONDITIONS

In this section, we expand further on a collapse solution being attracting and becoming independent from initial conditions. The solution for some time $\tau<0$ will contain the solution V_0 as well as a part dependent on initial conditions. The latter part of the solution will be shown to decay in collapsing coordinates as $\tau\rightarrow 0$.

We concentrate on the regime where r is finite and τ is small. This region can be thought of as the wake of the collapsing wave. The wave region is characterized by ξ being of order 1; thus, in the wake region, the quantity ξ is much larger than V_0, which is at most $O(1)$ everywhere. Applying this limit to the eigenfunction relation Eq. (2.11), it is seen that to leading order the terms containing V_0 become negligible. Equation (2.11) yields

$$\delta P=0, \quad (q+\alpha)\delta V+(1-\alpha)(\xi\cdot\nabla_{\xi}\delta V)=0. \tag{4.1}$$

Expressing this in scaled spherical coordinates (ρ, θ, ϕ) we find that (θ, ϕ) enter only as parameters and do not scale with the collapse. The scaled radial coordinate is denoted by ρ and scales as ξ in Eq. (2.3). This radial dependence can be separated and solved to yield

$$\delta V=\nu_n(\theta,\phi)\rho^{-n}, \tag{4.2}$$

where

$$n=\frac{q+\alpha}{1-\alpha} \tag{4.3}$$

is another separation constant, related to the growth rate q by $q=n(1-\alpha)-\alpha$. Note that for $0<\alpha<1$, if n is negative, then q is also. Following Eq. (2.10), we can construct the full eigenfunction as

$$V'=\nu_n(\theta,\phi)(-\tau)^{-q}\rho^{-n}. \tag{4.4}$$

The relevant modes for the wake region, those with negative n, then decay as $\tau\to 0$ since the associated q is also negative.

Translating back to the unscaled coordinates, we find that the eigenfunctions $(-\tau)^{-q}\delta V$ in the wake region become

$$v'\sim r^{-n}\nu_n(\theta,\phi) \tag{4.5}$$

for r_0 and v_0 zero. This can be understood as follows. Regions where the evolution is slow should be taken to be essentially stationary during the last stages of collapse. Thus Eq. (4.5) represents a Taylor series representation of the initial flow outside the collapse region. However, from the nature of the scaling transformation, the terms of this Taylor series are decaying eigenfunctions in the scaled coordinates. Loss of information occurs as $\tau\to 0$. Thus the initial flow has no influence on the evolution of the localized singularity. This is how information disappears as τ vanishes.

V. STABILITY OF BLOWUP SOLUTIONS

We next examine the stability of the steady solution (V_0,P_0) of the renormalized dynamical equation, Eqs. (2.5) and (2.6). Recall that the parameters of the steady solution (V_0,P_0) are r_0, t_0, v_0, and λ_0. We shall look at the time dependence of solutions with the transformation parameters slightly perturbed and under linearization.

To accomplish this, we introduce a transformation similar to (2.3), $(r,t,v,\Pi)\to(\tilde{\xi},\tilde{\tau},\tilde{V},\tilde{P})$, but with different parameters, denoted by the subscript 1.

$$\tilde{\xi}=\frac{r-r_1+v_1(t_1-t)}{\lambda_1(t_1-1)^{1-\alpha}},$$

$$\tilde{\tau}=t-t_1,$$

$$v=\frac{\lambda_1}{(t_1-t)^{\alpha}}\tilde{V}(\tilde{\xi},\tilde{\tau})+v_1,$$

$$\omega=\frac{1}{t_1-t}\nabla_{\xi}\times\tilde{V}(\tilde{\xi},\tilde{\tau}),$$

$$\Pi=\frac{\lambda_1^2}{(t_1-t)^{2\alpha}}\tilde{P}(\tilde{\xi},\tilde{\tau})+\frac{\lambda_1}{(t_1-t)^{\alpha}}v_1\cdot\tilde{V}(\tilde{\xi},\tilde{\tau}). \tag{5.1}$$

Now let $t_1=t_0+t'$, $v_1=v_0+v'$, $r_1=r_0+r'$, and $\lambda_1=\lambda_0+\lambda'$, and eliminate r,t,v between Eqs. (2.3) and (5.1). This gives

$$\tilde{V}=\frac{\lambda_0}{\lambda_0+\lambda'}\left(\frac{t'-\tau}{-\tau}\right)^{\alpha}V-\frac{(t'-\tau)^{\alpha}}{\lambda_0+\lambda'}v',$$

$$\xi=\frac{\lambda_0+\lambda'}{\lambda_0}\left(\frac{t'-\tau}{-\tau}\right)^{1-\alpha}\tilde{\xi}+\frac{r'-v_0t'-v'(t'-\tau)}{\lambda_0(-\tau)^{1-\alpha}},$$

$$\tau=\tilde{\tau}+t'. \tag{5.2}$$

Thus, if V and P is a solution of Eqs. (2.5) and (2.6), then $\tilde{V}(\tilde{\xi},\tilde{\tau},t',r',v',\lambda')$ and $\tilde{P}(\tilde{\xi},\tilde{\tau},t',r',v',\lambda')$ is also a solution of these equations. This follows since they both transform back to Eqs. (2.1) and (2.2). The two solutions are the same if the parameters (r',t',v',λ') are zero, and are nearby in a suitable normed space if they are small. In particular, a Taylor series in the parameters is term by term a solution.

If we take the solution with vanishing primed parameters to be the steady solution $V_0(\xi),P_0(\xi)$ of Eq. (2.5), then the nearby solutions are eigenfunctions generated by perturbations in the transformation parameters. As a first example, take all the primed parameters to vanish except t'. The leading terms in a Taylor series in t' are

$$\tilde{V}(\tilde{\xi},\tilde{\tau},t')=V_0(\xi)+t'\left(\frac{\partial\tilde{V}}{\partial t'}\right)_{t'=0}+\cdots. \tag{5.3}$$

Using Eq. (5.2) the derivative term becomes

$$\frac{\partial\tilde{V}}{\partial t'}=\frac{\partial}{\partial t'}\left[\left(\frac{t'-\tau}{-\tau}\right)^{\alpha}V_0\right]$$

$$=\frac{\alpha V_0}{-\tau}\left(\frac{t'-\tau}{-\tau}\right)^{\alpha-1}_{t'=0}+\frac{\partial\xi}{\partial t'}\cdot\frac{\partial V_0}{\partial\xi}$$

$$=\frac{1}{-\tau}\left[\alpha V_0+(1-\alpha)\xi\cdot\frac{\partial V_0}{\partial\xi}\right]. \tag{5.4}$$

Comparing to the expected form of the eigenfunctions in Eq. (2.10), $V(\xi,\tau)=V_0(\xi)+\epsilon(-\tau)^{-q}\delta V$, we find that the unstable eigenvalue is $q=1$ with the eigenvector $\alpha V_0+(1-\alpha)\xi\cdot\partial V_0/\partial\xi$. The solution is unstable to perturbations in the critical time. The amplitude of the instability ϵ is equal to t'. This is a logical result since the collapse solution has one particular critical time. Solutions formed with any transformation using a different critical time appear unstable.

Turning to perturbations in the spatial location of the singularity, we again expect an instability. The Taylor series for r' nonzero is

$$\tilde{V}(\tilde{\xi},\tilde{\tau},r')=V_0(\xi)+r'\cdot\left(\frac{\partial\tilde{V}}{\partial\tau'}\right)_{r'=0}+\cdots. \tag{5.5}$$

Again using Eq. (5.2) the derivative term becomes

$$\frac{\partial\tilde{V}}{\partial r'}=\frac{\partial\xi}{\partial r'}\cdot\frac{\partial V_0}{\partial\xi}=-\frac{1}{(-\tau)^{1-\alpha}}\frac{\partial V_0}{\partial\xi}. \tag{5.6}$$

The eigenvalue is $q=1-\alpha$ with the eigenvector $\partial V_0/\partial\xi$, which is a growing mode if $\alpha<1$.

Since λ is a scaling parameter, perturbations do not yield any time dependence. That is, $q=0$ and the perturbation is neutral, with eigenfunction given by

$$\frac{\partial\tilde{V}}{\partial\lambda'}=-V_0(\xi)+\xi\frac{\partial V_0}{\partial\xi}. \tag{5.7}$$

Thus we can expect that there is a one-parameter family of singular solutions, with the parameter being the size.

Finally, the response of the solution to perturbations in reference velocity should not be unstable due to Galilean invariance. Indeed for

$$\tilde{V}(\tilde{\xi},\tilde{\tau},v')=V_0(\xi)-v'\frac{(-\tau)^{\alpha}}{\lambda_0} \tag{5.8}$$

the derivative is

$$\frac{\partial\tilde{V}}{\partial v'}=\frac{(-\tau)^{\alpha}}{\lambda_0}\left(\frac{\partial V_0}{\partial\xi}+1\right). \tag{5.9}$$

Thus, the eigenvalue is $-\alpha$ and the eigenfunction is stable if α is positive.

We see that the stability of the solution to changes of the parameters is solely based on the form of the normalization, not of the governing equations.

VI. CONCLUSIONS

The purpose of this note is to discuss some general features of renormalization as applied to the problem of establishing the existence of singularities of Euler flow. In this method a general collapsing coordinate frame is introduced. If this coordinate frame can be adjusted until the evolution of a portion of the initial velocity field appears to be fixed, then the flow is singular. Then the question arises as to the stability of the flow that is fixed in the collapsing coordinates.

The existence of a flow breaks the continuous symmetries associated with the origin of time and space coordinates. We show the close relation of the localization of the singularity in time and space with the existence of instabilities of the flow. The instabilities only appear when there is some error in the estimated time and place of the singularity.

We also have identified large families of stable eigenfunctions. They provide a mechanism for independence of the singularity from details of the initial conditions. We also discuss the idea of an asymptotic self-similar solution. Conservation of energy, as a restriction on scaling, is ruled out.

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